Chapter 7. Intro to Counting

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7.12 Counting problem examples

Example: Assuming that a cell phone number is composed of area code: 3 digits, followed by the 7 digit number. How many different cell phone numbers are there that have at least one non-zero digit?

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Let **P** be the set of all cell phone numbers with at least one non-zero digit,
Then $\overline{\mathbf{P}}$ = the cell phone number with all zeros.

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$|\overline{\mathbf{P}}| = 1$ (0 0 0) 0 0 0 - 0 0 0 0

Therefore $|\mathbf{P}| = 9^{10} - 1 =$ 3, 486, 784, 400

7.7 Counting by complement

Counting by complement is a technique for counting the number of elements in a set **S** that *have a property* by counting the total number of elements in **S** and subtracting the number of elements in **S** that *do not have the property*.

The principle of counting by complement can be written using set notation where **P** is the subset of elements in **S** that *have the property*.

Example: Let's go back to the question about the password.
consider the following requirements for a password for an e-mail account:
- should consist of six to eight characters.
- each of these characters must be a digit or a letter of the alphabet (lower case or upper case)
- each password must contain at least one digit and at least one character.

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How many passwords are there?

Solution:
Let P be the total number of possible passwords of length $6 - 8$.
Let P_6 denote the number of possible passwords of length 6,
P_7 denote the number of possible passwords of length 7, and
P_8 denote the number of possible passwords of length 8.

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Solution:
Let P be the total number of possible passwords.
Let P_6 denote the number of possible passwords of length 6,

P_6: _ _ _ _ _ _

(lower case letters, upper case letters, digits: $26+26+10 = 62$)

$$62 \cdot 62 \cdot 62 \cdot 62 \cdot 62 \cdot 62 = 62^6$$

P_7 denote the number of possible passwords of length 7, and
P_8 denote the number of possible passwords of length 8.
By the sum rule $P = P_6 + P_7 + P_8$.

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Let P_6 denote the number of possible passwords of length *6*, 62^6

Recall restrictions: at least one digit and at least one character

Therefore we need to exclude those cases when it is purely digits and purely characters: 10^6 and 52^6. Therefore, $P_6 = 62^6 - 10^6 - 52^6$

P_7 denote the number of possible passwords of length *7*, and
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P_7 denote the number of possible passwords of length *7*, and

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P_8 denote the number of possible passwords of length *8*.

$P_8 = 62^8-10^8-52^8$

By the sum rule $P = P_6 + P_7 + P_8 = 62^6 - 10^6 – 52^6 + 62^7-10^7-52^7 + 62^8-10^8-52^8 =$

7.7 Counting by complement

$P = P_6 + P_7 + P_8 = 62^6 - 10^6 - 52^6 + 62^7 - 10^7 - 52^7 + 62^8 - 10^8 - 52^8$ = 56 800 235 584 – 1 000 000 – 19 770 609 664 + 3 521 614 606 208 – 10 000 000 – 1 028 071 702 528 + 218 340 105 584 896 – 100 000 000 – 53 459 728 531 456 = 167 410 838 583 040

7.8 Permutations with repetitions

Example: How many ways are there to scramble the letters in the word POLYNOMIAL?

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- There are 8 places left to put 2 Ls, then
- There are 6 places left to put 1 P, then
- There are 5 places left to put 1 Y, then
- There are 4 places left to put 1 N, then
- There are 3 places left to put 1 M, then
- There are 2 places left to put 1 I, and finally
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$$\binom{10}{2}\times\binom{8}{2}\times\binom{6}{1}\times\binom{5}{1}\times\binom{4}{1}\times\binom{3}{1}\times\binom{2}{1}\times\binom{1}{1}=\frac{10!}{2!(10-2)!}\times\frac{8!}{2!(8-2)!}\times\frac{6!}{1!(6-1)!}$$

$$\times\ \frac{5!}{1!(5-1)!}\times 4\times 3\times 2\times 1=\frac{10!}{2!2!}$$

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$$\binom{10}{2}\times\binom{8}{2}\times\binom{6}{1}\times\binom{5}{1}\times\binom{4}{1}\times\binom{3}{1}\times\binom{2}{1}\times\binom{1}{1}=\frac{10!}{2!\cancel{(10-2)!}}\times\frac{\cancel{8!}}{2!\cancel{(8-2)!}}\times\frac{\cancel{6!}}{1!\cancel{(6-1)!}}$$

$$\times\frac{\cancel{5!}}{1!\cancel{(5-1)!}}\times\cancel{4\times3\times2\times1}=\frac{10!}{2!2!}$$

7.8 Permutations with repetitions

Formula for counting permutations with repetition.

The number of distinct sequences with n_1 1's, n_2 2's, ..., n_k k's, where $n = n_1 + n_2 + \ldots + n_k$ is

$$\frac{n!}{n_1!\,n_2!\ldots n_k!}$$

7.9 Counting multisets

A set is a collection of distinct items.
A multiset is a collection that can have multiple instances of the same kind of item.

Example: {1, 1, 1, 2, 3, 4, 4} is a multiset because it contains three 1's and two 4's.

the order in which the elements are listed does not matter, so
{1, 1, 1, 2, 3, 4, 4} is equal to {1, 2, 3, 1, 4, 1, 4}.

Multisets are useful in modeling situations in which there are several varieties of objects and one can have multiple instances of the same variety.

7.9 Counting multisets

Example: Suppose that a customer at Dunking Doughnuts is selecting a half-dozen doughnuts to buy. There are three varieties: glazed, Boston cream, and chocolate glazed. Doughnuts of the same variety are indistinguishable, so one chocolate glazed doughnut is the same as any other chocolate glazed doughnut. Moreover, there is a good supply of each kind, so the Dunking Doughnuts location is in no danger of running out of any of the varieties. How many ways are there to select a set of 6 doughnuts?

An example of selection: 2 glazed, 2 Boston cream, 2 chocolate glazed
- a multiset

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Let ***n*** be the number of object to select, ***m*** be the number of varieties. We are not limited on the number of each variety available and objects of the same variety are indistinguishable.

Then the number of ways to select ***n*** objects from a set of ***m*** varieties is

$$\binom{n+m-1}{m-1} = \frac{(n+m-1)!}{(m-1)!(n+m-1-(m-1))!} = \frac{(n+m-1)!}{(m-1)!\,n!}$$

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An example of selection: 2 glazed, 2 Boston cream, 2 chocolate glazed - a multiset

$$\binom{n+m-1}{m-1}=\binom{6+3-1}{3-1}=\frac{(6+3-1)!}{(3-1)!6!}=\frac{8!}{2!6!}=\frac{8\times 7}{2}=28$$

$$\binom{n+m-1}{m-1}=\frac{(n+m-1)!}{(m-1)!(n+m-1-(m-1))!}=\frac{(n+m-1)!}{(m-1)!n!}$$

7.9 Counting multisets

Example: Suppose that 10 indistinguishable balls are to be placed into one of four bins. The bins are numbered, making them distinguishable. How many ways are there to place the balls in the bins?

$$\binom{n+m-1}{m-1}=\frac{(n+m-1)!}{(m-1)!(n+m-1-(m-1))!}=\frac{(n+m-1)!}{(m-1)!\,n!}$$

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Example: Suppose that 10 indistinguishable balls are to be placed into one of four bins. The bins are numbered, making them distinguishable. How many ways are there to place the balls in the bins?

balls: indistinguishable
bins: distinguishable

We will employ the same formula, where ***n*** = # of balls, ***m*** = # of bins

$$\binom{n+m-1}{m-1} = \binom{10+4-1}{4-1} = \frac{(10+4-1)!}{(4-1)!\,10!} = \frac{13!}{3!\,10!} = \frac{13\times 12\times 11}{3\times 2} = 286$$

$$\binom{n+m-1}{m-1} = \frac{(n+m-1)!}{(m-1)!\,(n+m-1-(m-1))!} = \frac{(n+m-1)!}{(m-1)!\,n!}$$

7.10 Assignment problems: Balls in bins

Many counting problems that ask about the number of ways to assign or distribute a set of items can be expressed abstractly by asking about the number of ways to place ***n*** balls into ***m*** different bins.

In all the problems presented in this material, the bins are numbered, so placing a ball in bin 1 is considered different than placing a ball in bin 2.

Some problems place different constraints on the number of balls that can be placed into the bins.

Problems also vary according to whether the balls are all the same (indistinguishable) or all different (distinguishable). If the balls are different, they are numbered 1 through n, and which ball gets placed in which bin matters.

7.10 Assignment problems: Balls in bins

	No restrictions	At most one ball per bin	Same number of balls in each bin
	(any positive ***m*** and ***n***)	(***m*** must be at least ***n***)	(***m*** must evenly divide ***n***)
Indistinguishable balls	$\binom{n+m-1}{m-1}$	$\binom{n}{m}$	1
Distinguishable balls	m^n	$P(m,n)$	$\left(\frac{n!}{(n/m)!^m}\right)$

The Inclusion – Exclusion Principle

(subtraction principle)

- suppose a task can be done in n_1 or in n_2 ways

(but some of the set of n_1 ways to do the task are the same as some of the n_2 ways to do the task)

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n_1+n_2-(# of ways to do the task in a way that is both among the set of n_1 ways and n_2 ways)

Example: How many bit strings of length eight that either start with a *1* bit or end with the two bits *11*?

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1. Let's count the ones that start with *1*:

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1. Let's count the ones that start with *1*:

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$1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7$

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3. Let's count the ones that start with *1* and end with *11*:

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Total: $2^7 + 2^6 - 2^5 = 128 + 64 - 32 = 160$

The Inclusion – Exclusion Principle

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Let A_1 and A_2 be sets.

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Let A_1 and A_2 be sets.
There are $|A_1|$ ways to select an element from A_1, and $|A_2|$ ways to select an element from A_2.

$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

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Example:
Every student in a discrete mathematics class is either a computer science or a mathematics major or is a joint major in these two subjects. How many students are is the class if there are *38* computer science majoes (including joint majors), *23* mathematics majors (including joint majors), and *7* joint majors?

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Answer: *38 + 23 – 7 = 54*