Example 1: Find $f(2)$, $f(3)$, $f(4)$ and $f(5)$ is $f$ is defined recursively by:

$$f(0) = f(1) = 1,$$
and

$$f(n+1) = f(n) / f(n-1), \quad \text{for } n = 1, 2, \ldots$$
Recursive Definitions and Structural induction practice

**Example 1:** Find $f(2), f(3), f(4)$ and $f(5)$ is $f$ is defined recursively by:

\[
\begin{align*}
  f(0) &= f(1) = 1, \text{ and} \\
  f(n+1) &= f(n) / f(n-1), \quad \text{for } n = 1, 2, \ldots
\end{align*}
\]

**Solution:**

\[
\begin{align*}
  f(2) &= f(1) / f(0) = 1 / 1 = 1 \\
  f(3) &= f(2) / f(1) = 1 / 1 = 1 \\
  f(4) &= f(3) / f(2) = 1 / 1 = 1 \\
  f(5) &= f(4) / f(3) = 1 / 1 = 1
\end{align*}
\]

**Answer:** $f(2) = f(3) = f(4) = f(5) = 1$
Example 2: Determine whether each of these proposed definitions is a valid recursive definition of a function $f: W \rightarrow Z$. If $f$ is well defined, find a non-recursive formula for $f(n)$ when $n \in W$.

a) $f(0) = 2$, $f(1) = 3$, $f(n) = f(n-2) - 2f(n-3)$ for $n \geq 2$

b) $f(0) = 0$, $f(1) = 1$, $f(n) = 2f(n-1)$ for $n \geq 2$
Example 2: Determine whether each of these proposed definitions is a valid recursive definition of a function $f: W \rightarrow Z$. If $f$ is well defined, find a non-recursive formula for $f(n)$ when $n \in W$.

a) $f(0) = 2, \quad f(1) = 3, \quad f(n) = f(n-2) - 2f(n-3)$ for $n \geq 2$

$f(2) = f(0) - 2f(-1)$, but $f$ is not defined on -1. Therefore, the definition is not valid.

b) $f(0) = 0, \quad f(1) = 1, \quad f(n) = 2f(n-1)$ for $n \geq 2$

$f(0) = 0, \quad f(1) = 1, \quad f(2) = 2f(1) = 2, \quad f(3) = 2f(2) = 4, \quad f(4) = 2f(3) = 8, \quad f(5) = 2f(4) = 16, \quad f(6) = 2f(5) = 32$

Answer: $f(0) = 0$, and $f(n) = 2^{n-1}$, for $n \geq 1$
Example 3: Give recursive definition of $S_m(n)$, the sum of the integer $m$ and the nonnegative integer $n$. 
Recursive Definitions and Structural induction practice

Example 3: Give recursive definition of $S_m(n)$, the sum of the integer $m$ and the nonnegative integer $n$.

Answer:

$S_m(0) = m$

$S_m(n+1) = S_m(n) + 1$, for all integers $n$

Let's check:

$S_3(4) = 3 + 4 = 7$

$S_3(4) = S_3(3) + 1 = (S_3(2) + 1) + 1 = S_3(2) + 2 = (S_3(1) + 1) + 2 = S_3(1) + 3 = (S_3(0) + 1) + 3 = S_3(0) + 4 = 3 + 4 = 7$
Example 4: Prove that $f_1^2 + f_2^2 + f_3^2 + \ldots + f_n^2 = f_n f_{n+1}$ for $n \in \mathbb{Z}^+$, where $f_n$ is the $n$th Fibonacci number.

Fibonacci numbers: $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, ...
Example 4: Prove that $f_1^2 + f_2^2 + f_3^2 + \ldots + f_n^2 = f_n f_{n+1}$ for $n \in \mathbb{Z}^+$, where $f_n$ is the $n$th Fibonacci number.

Fibonacci numbers: $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, ...

Proof (by math. Induction):
Basis step: $n = 1$  
$f_1^2 = 1^2 = 1$  
$f_1 f_2 = 1 \cdot 1 = 1$  
Therefore, $f_1^2 = f_1 f_2$

Inductive step: assume that for $k \in \mathbb{Z}^+$,  
$f_1^2 + f_2^2 + f_3^2 + \ldots + f_k^2 = f_k f_{k+1}$ (IH)

Let's show that in this case $f_1^2 + f_2^2 + f_3^2 + \ldots + f_k^2 + f_{k+1}^2 = f_{k+1} f_{k+2}$:

$f_1^2 + f_2^2 + f_3^2 + \ldots + f_k^2 + f_{k+1}^2 = f_k f_{k+1} + f_{k+1}^2 = f_{k+1} (f_k + f_{k+1}) = f_{k+1} f_{k+2}$

This completes the inductive step.

By math. Induction we proved the formula above.  
\[\text{q.e.d.}\]
Example 5: Give a recursive definition of the set of positive integer powers of 3.
Example 5: Give a recursive definition of the set of positive integer powers of 3.

Answer:

Basis step: \( 3 \in S \)
Recursive step: \( 3s \in S, \) if \( s \in S \)

Let's check:
\( 3, 3 \cdot 3 = 9, 9 \cdot 3 = 27, 27 \cdot 3 = 81, \ldots \)
\( 3^1, 3^2, 3^3, 3^4 \)
Example 6: Let $S$ be the subset of the set of ordered pairs of integers defined recursively by

Basis step: $(0,0) \in S$

Recursive step: if $(a,b) \in S,$
then $(a+2,b+3) \in S,$ and $(a+3,b+2) \in S$

a) List the elements of $S$ produced by the first five applications of the recursive definition.
Example 6: Let $S$ be the subset of the set of ordered pairs of integers defined recursively by

- **Basis step:** $(0,0) \in S$
- **Recursive step:** if $(a,b) \in S$,
  then $(a+2,b+3) \in S$, and $(a+3,b+2) \in S$

a) List the elements of $S$ produced by the first five applications of the recursive definition.

- $(0,0)$
- $(2,3), (3,2)$
- $(4,6), (5,5), (6,4)$
- $(6,9), (7,8), (8,7), (9,6)$
- $(8,12), (9,11), (10,10), (11,9), (12,8)$
Example 6: Let $S$ be the subset of the set of ordered pairs of integers defined recursively by

Basis step: $(0,0) \in S$

Recursive step: if $(a,b) \in S$,

then $(a+2,b+3) \in S$, and $(a+3,b+2) \in S$

b) Use strong induction on the number of applications of the recursive step of the def. to show that $5 \mid (a+b)$, if $(a,b) \in S$.

Let $P(n)$: “$5 \mid (a+b)$ if $(a+b)$ obtained by $n$ applications of the rec. step”
Example 6: Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis step: \((0,0) \in S\)
Recursive step: if \((a,b) \in S\),
then \((a+2,b+3) \in S\), and \((a+3,b+2) \in S\)

b) Use strong induction on the number of applications of the recursive step of the def. to show that \(5 \mid (a+b)\), if \((a,b) \in S\).

Let \(P(n): \text{“} 5 \mid (a+b) \text{ if } (a+b) \text{ obtained by } n \text{ applications of the rec. step”} \)

Basis step: \(P(0)\) is true, because \((0,0)\) is obtained by 0 steps and \(5\mid(0+0)\).
Inductive step: assume that \(5\mid(s+t)\), whenever \((s,t) \in S\) is obtained by \(k\) or fewer applications of the rec. step. (IH)

Let's consider an element obtained by \((k+1)\) applications of the rec. step: it was received from an element \((s,t) \in S\) obtained by \(k\) applications by either \((s+2,t+3)\) or \((s+3,t+2)\).

In the 1\(^{st}\) case \((s+2,t+3)\): \(s+2+t+3 = s+t+5\). By IH \(5\mid s+t\), and \(5\mid 5\), therefore \(5\mid s+t+5\)
In the 2\(^{nd}\) case \((s+3,t+2)\): \(s+3+t+2 = s+t+5\), so similarly to the previous reasoning \(5\mid s+t+5\)

This completes inductive step.
By strong induction we proved that \(5 \mid (a+b)\), if \((a,b) \in S\).
q.e.d.
Example 6: Let $S$ be the subset of the set of ordered pairs of integers defined recursively by

Basis step: $(0,0) \in S$

Recursive step: if $(a,b) \in S$,
then $(a+2,b+3) \in S$, and $(a+3,b+2) \in S$

c) Use structural induction to show that $5 \mid (a+b)$, if $(a,b) \in S$. 
Example 6: Let $S$ be the subset of the set of ordered pairs of integers defined recursively by

Basis step: $(0,0) \in S$
Recursive step: if $(a,b) \in S$,
then $(a+2,b+3) \in S$, and $(a+3,b+2) \in S$

c) Use structural induction to show that $5 \mid (a+b)$, if $(a,b) \in S$.
Basis step: $(0,0) \in S$: $0+0 = 0$, $0$ is divisible by $5$, i.e. $5 \mid 0$
Recursive/Inductive step: assume we get a pair $(s,t) \in S$ after $k \geq 1$ applications of rec. step from def. and $5 \mid s+t$. (IH)
Then for the next application of the rec. step there are two options:
Case 1: $(s+2,t+3) \in S$, in this case $(s+2)+(t+3) = s+t+5$
$5 \mid (s+t)$ (by IH), and $5 \mid 5$, therefore $5 \mid (s+t+5)$
Case 2: $(s+3,t+2) \in S$, in this case $(s+3)+(t+2) = s+t+5$
$5 \mid (s+t)$ (by IH), and $5 \mid 5$, therefore $5 \mid (s+t+5)$
This completes inductive step.
By structural induction we proved divisibility statement. q.e.d.